SUMS OF SOLID HORNED SPHERES

BY
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1. **Introduction.** In 1924 Alexander [2] described a 2-sphere in S^3 such that one complementary domain is not simply connected. This example has come to be known as Alexander's "horned sphere." If M denotes Alexander's horned sphere and U denotes the nonsimply connected component of $S^3 - M$, then the continuum $M \cup U = M^*$ is called Alexander's *solid horned sphere.*

Bing proved [5] that a continuum is homeomorphic with S^3 if it is the sum of three mutually exclusive sets M, U^1, and U^2 such that (a) there is a homeomorphism of $M \cup U^i$ $(i = 1, 2)$ onto Alexander's solid horned sphere that takes M onto the horned sphere and (b) there is a homeomorphism of $M \cup U^1$ onto $M \cup U^2$ that is pointwise fixed on M. We call this continuum $M \cup U^1 \cup U^2$ the *sum of two solid horned spheres sewn together along their boundaries with the identity homeomorphism*, and we use the notation $\Sigma(M^*, M^*)$ for this continuum.

Bing's paper answered a question raised by Wilder [14, Problem 4.6, p. 383], but Bing raised another question in the same paper relative to the same problem. Is the sum of two solid horned spheres homeomorphic with S^3 if it is not required that condition (b) in the paragraph above be satisfied? If condition (b) is not satisfied, we speak of two solid horned spheres sewn together along their boundaries with some homeomorphism of the horned sphere onto itself other than the identity. Casler showed [11] that the sum of two of Alexander's solid horned spheres sewn together with any homeomorphism of the horned sphere onto itself is S^3. Prior to Casler's work, Ball [4] defined a horned sphere and a homeomorphism of this sphere onto itself such that the continuum resulting from sewing two of his solid horned spheres together along their boundaries with this homeomorphism is different from S^3. It follows from Theorem 1 of this paper that if two of Ball's solid horned spheres are sewn together along their boundaries with the identity homeomorphism, the resulting continuum is topologically S^3.

We might note here that there are several questions about the structure of euclidean spaces that are closely related to questions about sums of solid horned spheres. Whenever the identity homeomorphism is used to sew two solid horned

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spheres together and the sum is homeomorphic with S^3, there is an associated embedding of a 2-sphere in S^3 which is the fixed point set of a homeomorphism of period 2 of S^3 onto itself. There is also an upper semicontinuous decomposition of S^3 which may be associated with the sum of two solid horned spheres. We will discuss specific questions about upper semicontinuous decompositions of S^3 later in this paper.

In this paper we give a general definition for a horned sphere and investigate some conditions under which the sum of two solid horned spheres sewn together along their boundaries with the identity homeomorphism is homeomorphic with S^3. We will follow the general procedure used by Bing and Casler in the papers mentioned above and work with decomposition spaces associated with the sum of the solid horned spheres.

2. **Notation and definitions.** Suppose K is a polyhedral cube in E^3 containing m $(m > 1)$ disjoint polyhedral cubes $K_1, K_2, \cdots, K_m$ in its interior. In the interior of each K_i are n_i $(n_i > 1)$ disjoint polyhedral cubes $K_{i1}, K_{i2}, \cdots, K_{in_i}$. In the same manner, there are more disjoint polyhedral cubes in the interior of each K_{ij}. We continue this process indefinitely and require that each cube at the nth stage $(n > 0)$ have diameter less than $1/n$. Let H_n be the union of all polyhedral cubes at the nth stage (with $H_0 = K$). The intersection of the nested sequence of closed sets $H_0, H_1, \cdots$ is a tame Cantor set [8]. We will be concerned with the number of cubes at each stage of such a construction, and it will be useful in this connection to introduce some notation suggested by Casler in [11]. An *admissible n-sequence* for such a construction is an n-termed sequence which is a subscript for one of the polyhedral cubes of the nth stage of the construction. We denote an admissible n-sequence by $n\alpha$. An admissible m-sequence $m\alpha$ is said to *follow* an admissible n-sequence $n\alpha$ if $m > n$ and $n\alpha$ is the subsequence of $m\alpha$ consisting of the first n terms of $m\alpha$. Thus $m\alpha$ follows $n\alpha$ if and only if $K_{m\alpha}$ is a proper subset of $K_{n\alpha}$. We use $n\alpha i$ to denote the $(n+1)$-admissible sequence following $n\alpha$ whose last term is i. We use $\{A_{n\alpha}\}$ to denote the collection of all sets $A_{n\alpha}$ where $n\alpha$ is an admissible n-sequence, and we use $\Sigma A_{n\alpha}$ to denote the sum of all elements of $\{A_{n\alpha}\}$. It is also useful at times to use 0α with a set not having a subscript; thus $K_{0\alpha} = K$. In this notation, the tame Cantor set mentioned above is given by $\bigcap_{n=0}^{\infty}(\Sigma K_{n\alpha})$.

To illustrate, we may take as an example the ordinary "middle-third" construction for a Cantor set in which there are two cubes of a given stage in each cube of the preceding stage. Then each term of an admissible sequence for this construction is 1 or 2. For this construction 2α would denote one of the four admissible 2-sequences 11, 12, 21, 22. If 5α were the admissible 5-sequence 12212, then $5\alpha i$ would be one of the two sequences 122121, or 122122.

We note that while a given construction for a tame Cantor set determines a set of admissible sequences, a given set of admissible sequences also determines the construction of a tame Cantor set. In the definition of a horned sphere that follows we shall assume as given a set of admissible sequences with corresponding collections $\{K_{n\alpha}\}$ defining a Cantor set M_0.

We use S_0 to denote the polyhedral 2-sphere which is the boundary of the cube K. Another polyhedral 2-sphere S_1 obtained from S_0 by replacing a finite number of disks on S_0 by other disks in K which we call *horns*. A horn is the union of a polyhedral annulus T (which Bing [5] called the "surface of a tube") and a polyhedral disk D which we call the *end* of the horn. The ends of the horns from S_0 are disks on the boundaries of the $K_{1\alpha}$, and we require that each $K_{1\alpha}$ in $\{K_{1\alpha}\}$ contain the end of at least one horn. The boundary components of the annulus T are $\mathrm{Bd}\, D$ and the boundary of a disk on S_0. The horns are pairwise disjoint and, for each horn $T \cup D$, we have $\mathrm{Int}\, T \subset (\mathrm{Int}\, K - \Sigma K_{1\alpha})$.

In a similar manner, we obtain a polyhedral 2-sphere S_2 by replacing disks in the ends of the horns of S_1 by horns in $\Sigma K_{1\alpha}$ having their ends in $\Sigma K_{2\alpha}$. Again we require that the horns be pairwise disjoint, that at least one horn be run to each $K_{2\alpha}$ in $\{K_{2\alpha}\}$, and that no horn intersect $(\Sigma \mathrm{Bd}\, K_{1\alpha} \cup \Sigma K_{2\alpha})$ except in its end and boundary. This process may be continued to obtain a sequence of polyhedral 2-spheres $\{S_n\}$. Generally, S_{n+1} is obtained from S_n by running horns in $\Sigma K_{n\alpha}$ from the ends of horns of S_n to elements of $\{K_{(n+1)\alpha}\}$.

We require that the construction described above be done so that the limit of the sequence $\{S_n\}$ is a 2-sphere M. This limiting 2-sphere we call a *horned sphere*. If we denote the bounded component of $E^3 - M$ by U, then $M^* = M \cup U$ is called a *solid horned sphere*, and $\Sigma(M^*, M^*)$ denotes the *sum of two solid horned sphere sewn together along their boundaries with the identity homeomorphism*.

Note that this definition of a horned sphere includes many embeddings of 2-spheres in E^3. For example, if only one horn is run to each cube at each stage, then M is tame because the boundaries of the $K_{n\alpha}$ provide spanning disks which satisfy Burgess' characterization of tame surfaces in E^3 [10]. If the horns are not properly intertwined, it is also possible to obtain horned spheres which are tamely embedded in E^3 even though each cube at each stage has more than one horn running to it. By our definition, a horned sphere is always locally tame except on the Cantor set M_0. We have pictured a portion of the construction of a horned sphere in Figure 1. Other horned spheres are indicated in Figures 5 and 12 of this paper and in [4], [5], and [11].

There are questions that may be raised about the different embeddings of various horned spheres. For example, as far as we have been able to ascertain, it was not known until Casler obtained his result [11] that Alexander's horned sphere and Ball's horned sphere [4] are not equivalently embedded in S^3. We do not consider this general problem here although some conclusions can be drawn about special cases from the work presented here. For example, it will follow from §4 and §5 that the horned spheres M_{I} and M_{II} are not equivalently embedded in S^3.

A k-horned sphere is a 2-sphere which is the limit of a sequence $\{S_n\}$ in which k horns are run in a specified way, as indicated in the cases below, from S_{n-1} to each $K_{n\alpha}$ in $\{K_{n\alpha}\}$, $(n = 1, 2, \cdots)$. In this paper we confine our attention to two particular

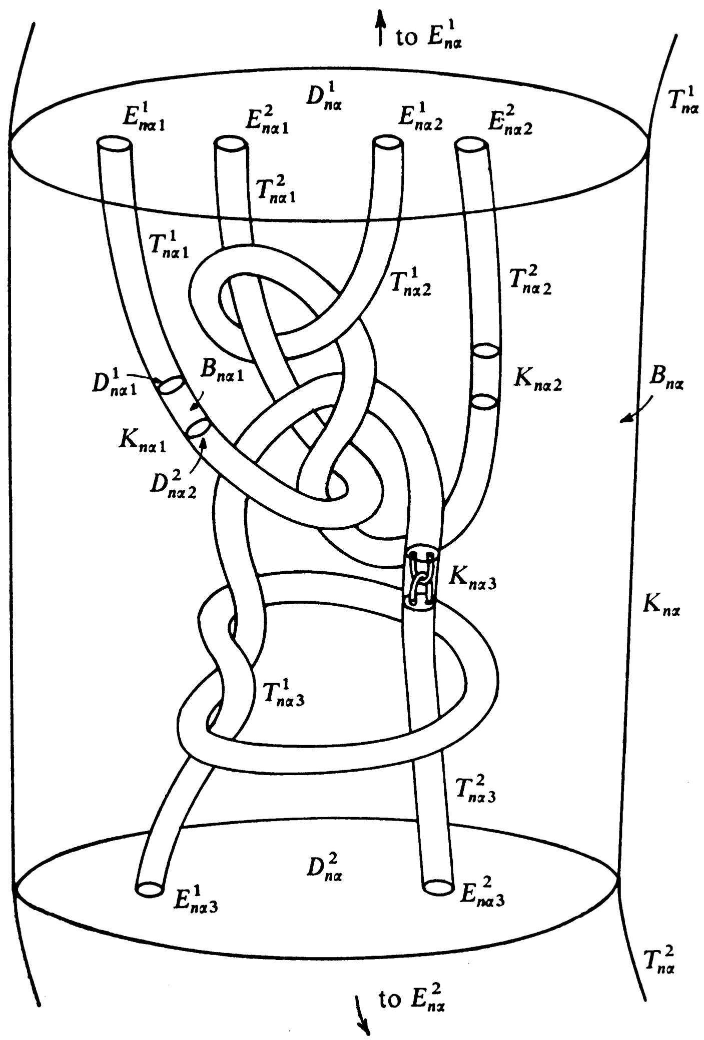

FIGURE 1

types of horned spheres, 2-horned and 3-horned spheres. Alexander's and Ball's horned spheres are 2-horned spheres. Figure 1 also shows part of a typical 2-horned sphere. Figures 5 and 12 depict 3-horned spheres. In §3, we prove that if M^* is any solid 2-horned sphere, then $\Sigma(M^*, M^*)$ is homeomorphic with S^3. In §4 and §5, we examine two solid 3-horned spheres M^*_{I} and M^*_{II}. The first of these is a 3-horned analogue of Alexander's solid 2-horned sphere and has the property that $\Sigma(M_{\rm I}^*, M_{\rm I}^*)$ is homeomorphic with S^3. For $M_{\rm II}^*$, however, $\Sigma(M_{\rm II}^*, M_{\rm II}^*)$ is topologically different from S^3.

Our general method of attack will be to consider S^3 as the one-point compactification of E^3, for which we write $E^3 + p$. We define an upper semi-continuous decomposition of $E^3 + p$ such that the associated decomposition space is homeomorphic with the sum of two of the solid horned spheres under consideration. We will then work with the decomposition space and show that it is or is not homeomorphic with $E^3 + p$. Since our methods and notation are similar to those used by Bing [5] and Casler [11], it would be helpful to be familiar with their work.

3. **2-horned spheres.** In order to work with 2-horned spheres it is necessary to introduce more specific notation than we used in §2 for the general definition of a horned sphere. We have a Cantor set defined by $M_0 = \bigcap_{n=0}^{\infty}(\Sigma K_{n\alpha})$. For convenience we assume that each $K_{n\alpha}$ is a solid right circular cylinder. The boundary of $K_{n\alpha}$ consists of two bases, disjoint disks $D_{n\alpha}^1$ and $D_{n\alpha}^2$, and an annulus $B_{n\alpha}$ whose boundary components are $\operatorname{Bd} D_{n\alpha}^1$ and $\operatorname{Bd} D_{n\alpha}^2$. The disks $D_{n\alpha}^1$ and $D_{n\alpha}^2$ will be used as ends of horns in the construction of the horned sphere. We will use $E_{n\alpha}^1$ and $E_{n\alpha}^2$ to denote the disjoint disks replaced by the horns running to $K_{n\alpha}$. The $E_{n\alpha}^i$ occur in pairs in $\operatorname{Int} D_{(n-1)\alpha}^i$, and each $D_{(n-1)\alpha}^i$ contains at least two $E_{n\alpha}^i$ in its interior. The horns then are the sets $T_{n\alpha}^i \cup D_{n\alpha}^i$ $(i = 1,2)$ where the boundary components of the annulus $T_{n\alpha}^i$ are $\operatorname{Bd} E_{n\alpha}^i$ and $\operatorname{Bd} D_{n\alpha}^i$. The construction described here is indicated in Figure 1. It should be noticed that while we require that exactly two horns end in each $K_{n\alpha}$, there is no restriction on the number of cubes in the finite collection $\{K_{n\alpha}\}$.

The sequence of 2-spheres whose limit is the 2-horned sphere M is defined as follows:

$$S_0 = \operatorname{Bd} K, \text{ and, for } n \geqq 1,$$
$$S_n = (S_{n-1} - \Sigma E_{n\alpha}^i) \cup (\Sigma(T_{n\alpha}^i \cup D_{n\alpha}^i)),$$

where the indicated summation is taken over all admissible n-sequences and $i = 1,2$. As in §2, we let U denote the bounded complementary domain of M in E^3, and call $M^* = M \cup U$ a solid 2-horned sphere. Our goal in this section is the proof of the following theorem:

THEOREM 1. *If M^* is a solid 2-horned sphere, then $\Sigma(M^*, M^*)$ is homeomorphic with S^3.*

As indicated at the end of §2, we will defer the proof of Theorem 1 until we have some preliminary results.

Let P denote the plane in E^3 with equation $y = 0$. We assume that P contains the ideal point p and that the closure operation is taken relative to $E^3 + p$. We use L and R to denote the left and right complementary domains of P where our point of view is that of an observer located on the positive x-axis looking toward the origin. For any set H in E^3 we use the notation LH for $\mathrm{Cl}(L) \cap H$ and RH for $\mathrm{Cl}(R) \cap H$. It is also useful to let r denote the reflection of E^3 onto itself about the plane P; that is, $r(x, y, z) = (x, -y, z)$.

When Bing sewed two Alexander solid horned spheres together [5], he described a certain collection of solid tori in E^3 and used them to define a homeomorphism between $\Sigma(M^*, M^*)$ and his decomposition space. To follow the same procedure, we must first define some solid tori. To do this, we use an unknotted polyhedral solid torus A which is symmetric with respect to the plane P and such that $A \cap P$ is the sum of a pair of mutually exclusive disks. We then introduce a sequence of homeomorphisms $\{f_n\}$ which are used only to define collections $\{A_{n\alpha}\}$ of solid tori for all admissible n-sequences.

Since LA is topologically a solid cylinder, there is a piecewise linear homeomorphism f_1 of K onto LA which takes the bases of K onto the two components of $A \cap P$ and which preserves cross sections of K. Then for each admissible 1-sequence 1α, the image of the annulus $(T^1_{1\alpha} \cup B_{1\alpha} \cup T^2_{1\alpha})$ under f_1 together with the image of the same annulus under rf_1 bounds a solid torus in $\mathrm{Int}\, A$ which we denote by $A_{1\alpha}$. Each $A_{1\alpha}$ is symmetric with respect to P, and $A_{1\alpha} \cap P$ is the sum of the two mutually exclusive disks $f_1(E^1_{1\alpha})$ and $f_1(E^2_{1\alpha})$. Similarly there is a homeomorphism f_2 of $\Sigma K_{1\alpha}$ onto $\Sigma LA_{1\alpha}$ which takes each $K_{1\alpha}$ onto the corresponding $LA_{1\alpha}$ in such a way that cross sections of $K_{1\alpha}$ are preserved and $f_2(D^i_{1\alpha}) = f_1(E^i_{1\alpha})$, $(i = 1, 2)$. Then the images of the annulus $(T^1_{2\alpha} \cup B_{2\alpha} \cup T^2_{2\alpha})$ under f_2 and rf_2 bound a solid torus $A_{2\alpha}$ which is symmetric with respect to P, and $A_{2\alpha} \cap P = f_2(E^1_{2\alpha}) \cup f_2(E^2_{2\alpha})$. If 2α follows 1α, then $A_{2\alpha}$ is a subset of $\mathrm{Int}\, A_{1\alpha}$. This process may clearly be continued to define the finite collections $\{A_{n\alpha}\}$ for all positive integers n. We denote $\bigcap_{n=0}^{\infty} (\Sigma A_{n\alpha})$ by A_0 and let W be the upper semicontinuous decomposition of $E^3 + p$ consisting of the components of A_0 and the points of $(E^3 + p) - A_0$.

We may take the f_n to be such that the $A_{n\alpha}$ are polyhedral and "smooth" near P in the following sense. There is a positive number δ such that for any $n\alpha$, $A_{n\alpha} \cap (P \times [-\delta, \delta]) = (A_{n\alpha} \cap P) \times [-\delta, \delta]$. We may also suppose that the maximum cross sectional diameter of the $A_{n\alpha}$ occurs in $A_{n\alpha} \cap P$. See Figure 2 for a picture of the construction of the $A_{n\alpha}$ corresponding to the stage of the construction of M shown in Figure 1. We have not attempted to indicate any knotting of $A_{n\alpha}$ in $E^3 + p$.

Having defined the $A_{n\alpha}$ we may follow Bing's procedure [5]. We let $M_n = S_n - \Sigma D^i_{n\alpha}$ and $U_n = U - \Sigma K_{n\alpha}$. Thus M_n is a 2-sphere minus a finite collection of disks, and the boundary of U_n is $M_n \cup \Sigma B_{n\alpha}$. There is a homeomorphism F_1 of $\mathrm{Cl}(U_1)$ onto $\mathrm{Cl}(L) - \Sigma \mathrm{Int}\, A_{1\alpha}$ such that $F_1(M_1) = P - \Sigma A_{1\alpha}$ and $F_1(B_{1\alpha}) = \mathrm{Cl}(L) \cap \mathrm{Bd}\, A_{1\alpha}$. Setting $G_1 = rF_1$, we obtain a homeomorphism of $\mathrm{Cl}(U_1)$ onto $\mathrm{Cl}(R) - \Sigma \mathrm{Int}\, A_{1\alpha}$ such that $G_1 | M_1 = F_1 | M_1$ and $G_1(B_{1\alpha}) = \mathrm{Cl}(R) \cap \mathrm{Bd}\, A_{1\alpha}$. Generally, there is a homeomorphism F_n of $\mathrm{Cl}(U_n)$

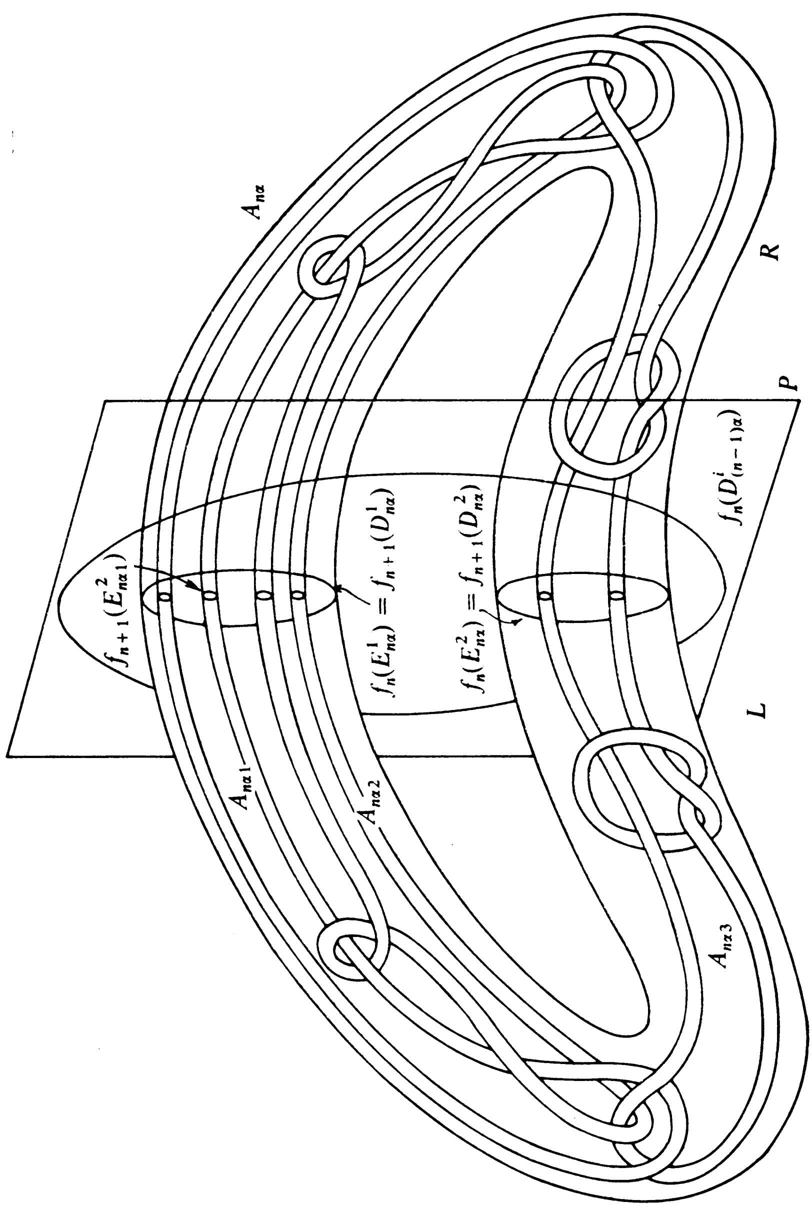
$A_{n\alpha}$
R
P
$f_n(D^i_{(n-1)\alpha})$
$f_{n+1}(E^2_{n\alpha 1})$
$f_n(E^1_{n\alpha}) = f_{n+1}(D^1_{n\alpha})$
$f_n(E^2_{n\alpha}) = f_{n+1}(D^2_{n\alpha})$
L
$A_{n\alpha 1}$
$A_{n\alpha 2}$
$A_{n\alpha 3}$

FIGURE 2

onto $\mathrm{Cl}(L) - \Sigma \operatorname{Int} A_{n\alpha}$ such that F_n agrees with F_{n-1} on $\mathrm{Cl}(U_{n-1})$, $F_n(M_n) = P - \Sigma A_{n\alpha}$, and $F_n(B_{n\alpha}) = \mathrm{Cl}(L) \cap \mathrm{Bd}\, A_{n\alpha}$. We set $G_n = rF_n$. The sequences $\{F_n\}$ and $\{G_n\}$ converge to homeomorphisms F and G of $M^* - M_0$ onto $\mathrm{Cl}(L) - A_0$ and $\mathrm{Cl}(R) - A_0$, respectively. The limit homeomorphisms F and G define a homeomorphism of $\Sigma(M^*, M^*)$ onto the decomposition space $(E^3 + p)/W$.

To show that the decomposition space is homeomorphic with $E^3 + p$, we shrink the components of A_0 to points by repeating a process of shrinking the tori $\{A_{n\alpha}\}$ defining A_0. For notational purposes, we may think of each solid torus $A_{n\alpha}$ as the image under a homeomorphism g of the Cartesian product of a disk D and a circle C. We parameterize the circle by identifying the endpoints of the unit interval. Then a point of $D \times C$ may be written (x, t) where x is a point of D and t is a real number in the interval $[0, 1]$. A *cross-section* of $A_{n\alpha}$ is a disk $g(D \times t)$ for some t between 0 and 1. We sometimes refer to $g(D \times t)$ as the *t-cross-section* of $A_{n\alpha}$. We require that each component of $A_{n\alpha} \cap P$ be a cross section of $A_{n\alpha}$ and, as mentioned above, that no cross section have diameter greater than the diameter of either component of $A_{n\alpha} \cap P$. A *section* of $A_{n\alpha}$ is the closure of the subset of $A_{n\alpha}$ between two cross sections of $A_{n\alpha}$. Each pair of cross sections of $A_{n\alpha}$ determines two sections of $A_{n\alpha}$.

An important tool in the shrinking process to follow is a homeomorphism of a type described in Lemma 1.1. We call such a homeomorphism a *fundamental homeomorphism*. It is similar to Casler's "basic homeomorphism" [11, Lemma 3], adapted for solid tori rather than solid cylinders. Intuitively, a fundamental homeomorphism may be thought of as sliding a given set in the interior of a solid torus along the central axis of the torus with some attendant "squeezing" and "stretching." We make this intuitive notion more precise in the statement of the lemma, but we do not give a proof for the lemma.

LEMMA 1.1. *Suppose X is a closed subset of* $\operatorname{Int} A_{n\alpha}$ *and Y_1 and Y_2 are two sections of $A_{n\alpha}$. Then there is a homeomorphism h of $A_{n\alpha}$ onto itself such that*

(1) *h is the identity on* $\mathrm{Bd}\, A_{n\alpha}$,

(2) *h takes the portion of X in any cross-section of $A_{n\alpha}$ into a cross-section of $A_{n\alpha}$, and*

(3) *h takes* $(X \cap \operatorname{Int} Y_1)$ *into* $\operatorname{Int} Y_2$.

Furthermore, h may be taken to be the identity on any given section of $A_{n\alpha}$ in $\mathrm{Cl}(A_{n\alpha} - (Y_1 \cup Y_2))$.

LEMMA 1.2. *For any $\varepsilon > 0$ and any positive integer j, there is a positive integer s and a homeomorphism of $E^3 + p$ onto itself which is fixed on $(E^3 + p) - \Sigma A_{j\alpha}$ and which takes each component of $\Sigma A_{s\alpha}$ onto a set of diameter less than ε.*

Proof. There is an integer n, larger than j, such that the diameter of the largest cross section of any $A_{n\alpha}$ is less than $\varepsilon/3$. We will describe a homeomorphism for a given element $A_{n\alpha}$ of $\{A_{n\alpha}\}$ such that the homeomorphism is fixed on the boundary of $A_{n\alpha}$ and takes all the solid tori of some later stage in $A_{n\alpha}$ onto sets of diameter less than ε. Since there are only finitely many $A_{n\alpha}$, it is clear that if we can get such a homeomorphism for one $A_{n\alpha}$, we can get a homeomorphism satisfying the conclusions of the lemma.

Each cross section of $A_{n\alpha}$ has diameter less than $\varepsilon/3$, so there is a finite collection of cross sections of $A_{n\alpha}$ which partition $A_{n\alpha}$ into an even number of sections, $C_1, C_2, \cdots, C_{2k}$, numbered consecutively around $A_{n\alpha}$, such that each C_i has diameter less than $\varepsilon/3$. This can be done so that C_1 is in L and C_{k+1} is in R. We call C_1 and and C_{k+1} the *left end-section* and *right end-section*, respectively, of $A_{n\alpha}$. Let

$$P_1 = C_1 \cap (C_2 \cup C_{2k}),\ P_2 = (C_2 \cap C_3) \cup (C_{2k} \cap C_{2k-1}),$$

$$P_3 = (C_3 \cap C_4) \cup (C_{2k-1} \cap C_{2k-2}), \cdots, P_k = C_{k+1} \cap (C_k \cup C_{k+2}).$$

Thus each P_i is the sum of a pair of cross sections of $A_{n\alpha}$, and if a connected subset of $A_{n\alpha}$ intersects no more than one P_i, the subset must have diameter less than ε. See Figure 3, where no knotting of a $A_{n\alpha}$ in E^3 is shown because any such knotting is immaterial for this lemma.

Let P' be a plane in L parallel to P and such that the distance between P' and P is less than δ, where δ is the number identified previously in our description of the sets $A_{n\alpha}$. Let X be the sum of all elements of $\{A_{(n+1)\alpha}\}$ in $A_{n\alpha}$. Let Y_2 be the section C_1, and let Y_1 be the section of $A_{n\alpha}$ to the left of P'. By Lemma 1.1 there is a homeomorphism which is fixed to the right of P and which takes the portion of X in $\operatorname{Int} Y_1$ into $\operatorname{Int} Y_2$. Similarly, there is a plane $P'' = r(P')$ in R and a fundamental homeomorphism, fixed to the left of P which takes the portion of X to the right of P'' into the right end-section C_{k+1}. We use h_0 to denote the composition of these two fundamental homeomorphisms, extended as the identity outside $A_{n\alpha}$ to a homeomorphism of $E^3 + p$ onto itself. The effect of h_0 is to move all the linking and knotting of $\Sigma A_{m\alpha} (m > n)$ that may occur in $A_{n\alpha}$ out into the two end-sections of $A_{n\alpha}$. For any component $A_{m\alpha}$ of $(\Sigma A_{m\alpha}) \cap A_{n\alpha}$ $(m > n)$, $h_0(A_{m\alpha})$ intersects each cross section of $A_{n\alpha}$ between P_1 and P_k in the sum of two disjoint disks. See Figure 3.

Lemma 1.2 now follows by repeated applications of Lemma 1.1. We describe a finite sequence of homeomorphisms, each of which is a composition of pairs of fundamental homeomorphisms extended to $E^3 + p$. Let h_1 be a homeomorphism of $E^3 + p$ onto itself which agrees with h_0 outside $h_0(A_{n\alpha})$, which is fixed on $(\bigcup_{i=3}^{k} C_i) \cup (\bigcup_{i=k+3}^{2k} C_i)$, and which takes the portion of $h_0(X)$ in C_1 into C_2 and the portion of $h_0(X)$ in C_{k+1} into C_{k+2}. Then, for each $A_{(n+1)\alpha}$ in $A_{n\alpha}$, $h_1 h_0(A_{(n+1)\alpha})$ intersects at most one of P_1 and P_k. Compare Figure 4 with Figure 3.

If $h_1 h_0(A_{(n+1)\alpha})$ intersects P_1 and not P_k, we may select the section of $h_1 h_0(A_{(n+1)\alpha})$ in C_2 for Y_1 and one of the sections of $h_1 h_0(A_{(n+1)\alpha})$ in C_{2k} for Y_2 and apply a fundamental homeomorphism f_2 to $h_1 h_0(A_{(n+1)\alpha})$ which is fixed to the right of P_2 and which moves the portion of $h_1 h_0(\Sigma A_{(n+2)\alpha})$ in $\operatorname{Int} Y_1$ into

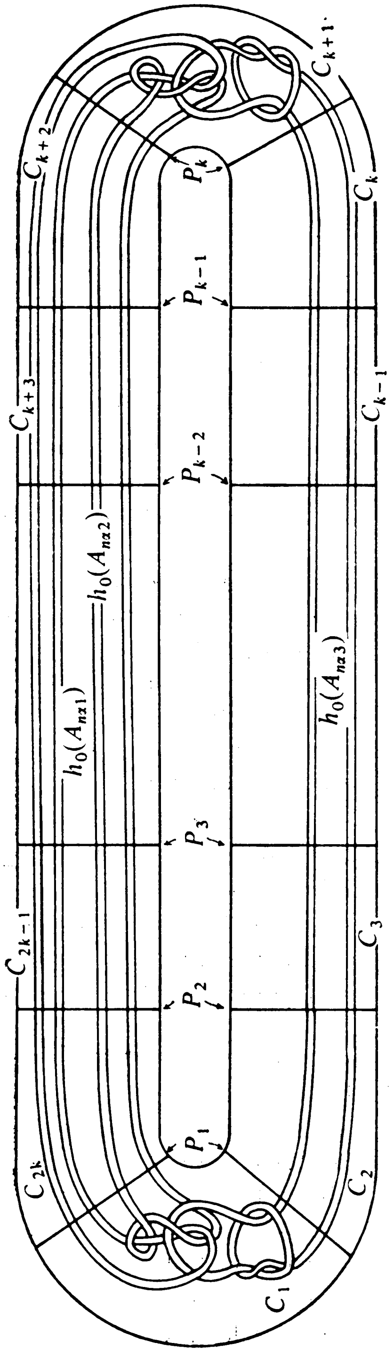

FIGURE 3

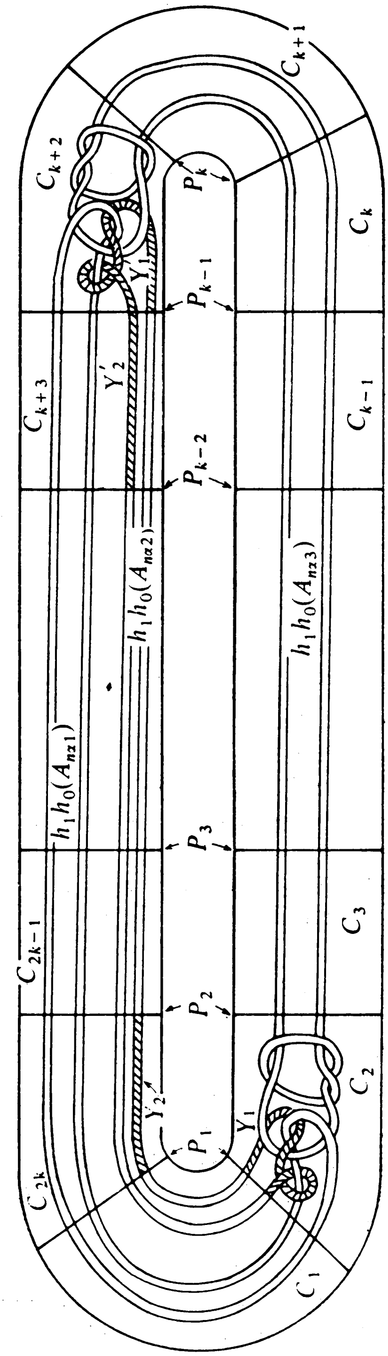

FIGURE 4

Int Y_2. Similarly we may take the section of $h_1h_0(A_{(n+1)\alpha})$ in C_{k+2} for Y_1' and a section of $h_1h_0(A_{(n+1)\alpha})$ in C_{k+3} for Y_2' and apply another fundamental homeomorphism f_2' to $f_2h_1h_0(A_{(n+1)\alpha})$. Care must be taken to select Y_2' in the same direction from Y_1' around $h_1h_0(A_{(n+1)\alpha})$ as Y_2 is from Y_1. One appropriate choice for Y_1, Y_2, Y_1', Y_2' is shaded in Figure 4. Then for any $A_{(n+2)\alpha}$ in $A_{(n+1)\alpha}$, $f_2'f_2h_1h_0(A_{(n+2)\alpha})$ intersects at most one of the sets P_1 and P_{k-1}.

If $h_1h_0(A_{(n+1)\alpha})$ intersects P_k and not P_1, there is a pair of fundamental homeomorphisms f_2 and f_2' such that for any $A_{(n+2)\alpha}$ in $A_{(n+1)\alpha}$, $f_2'f_2h_1h_0(A_{(n+2\alpha)})$ intersects at most one of P_2 and P_k. There is a similar pair of homeomorphisms for each $A_{(n+1)\alpha}$ in $A_{n\alpha}$. Let h_2 denote the homeomorphism of E^3+p onto itself which agrees with h_1h_0 outside $h_1h_0(\sum A_{(n+1)\alpha})\cap A_{n\alpha}$ and which agrees with the appropriate homeomorphism $f_2'f_2$ on each component of $h_1h_0(\sum A_{(n+1)\alpha})$ in $A_{n\alpha}$. Then for each $A_{(n+2)\alpha}$ in $A_{n\alpha}$, $h_2h_1h_0(A_{(n+2)\alpha})$ intersects no more than $k-2$ consecutive P_i.

We may continue in this fashion to define a sequence of homeomorphisms $h_1, h_2, \cdots, h_{k-1}$. If we let h denote the homeomorphism $h_{k-1}h_{k-2}\cdots h_1h_0$ of E^3+p onto itself, and $s = n+k-1$, then h is the homeomorphism mentioned in the first paragraph of the proof of this lemma.

We have established that $\Sigma(M^*, M^*)$ is homeomorphic with the decomposition space $(E^3+p)/W$. To complete the proof of Theorem 1 it is sufficient to show that the decomposition space is homeomorphic with E^3+p. This follows from Lemma 1.2 in exactly the same way that Bing's theorem follows from his lemma [5, pp. 359–361].

4. **The 3-horned sphere M_{I}.** For this construction we again let the tame Cantor set M_0 be defined by $\bigcap_{n=0}^{\infty}(\sum K_{n\alpha})$, where the $K_{n\alpha}$ are polyhedral cubes as described in §2. We use $C_{n\alpha}$ for the boundary of $K_{n\alpha}$. Each $K_{n\alpha}$ contains three disjoint cubes $K_{n\alpha 1}, K_{n\alpha 2}$, and $K_{n\alpha 3}$ so that the range for each term of an admissible sequence is the set $\{1,2,3\}$.

Let $D_{n\alpha 1}$, $D_{n\alpha 2}$, and $D_{n\alpha 3}$ be three mutually exclusive disks on $C_{n\alpha}$, and let $E_{n\alpha i1}$ $E_{n\alpha i2}$, and $E_{n\alpha i3}$ be three mutually exclusive disks in Int $D_{n\alpha i}$, $(i = 1,2,3)$. Similarly each $C_{n\alpha i}$ contains mutually exclusive disks $D_{n\alpha i1}$, $D_{n\alpha i2}$, and $D_{n\alpha i3}$, $(i = 1,2,3)$. Let $T_{n\alpha ij}$ be a polyhedral annulus in $K_{n\alpha}$ with boundary components Bd $D_{n\alpha ij}$ and Bd $E_{n\alpha ij}$, $(i,j = 1,2,3)$. The embedding of $(\bigcup_{i,j=1}^{3} T_{n\alpha ij}) \cup (\bigcup_{i=1}^{3} C_{n\alpha i})$ in $K_{n\alpha}$ is indicated in Figure 5.

We define the sequence of 2-spheres $\{S_n\}$ by

$$S_0 = C, \quad \text{and,} \quad \text{for} \quad n \geqq 1,$$

$$S_n = (S_{n-1} - \sum E_{(n+1)\alpha}) \cup (\sum (T_{(n+1)\alpha} \cup D_{(n+1)\alpha})).$$

As in the 2-horned case, the 3-horned sphere M_{I} is the limiting set of $\{S_n\}$, and

$M_{\text{I}}^* = M_{\text{I}} \cup U$ is the solid 3-horned sphere. We also use M_n and U_n with the same meaning as in §3.

We define an upper semi-continuous decomposition of $E^3 + p$ in almost exactly the same way for the 3-horned sphere as we did for the 2-horned sphere. We use P, L, and R as before for the xz-plane and its complementary domains. For this decomposition, A is a solid double torus, symmetric with respect to P, such that LA is a neighborhood in $\mathrm{Cl}(L)$ of a triod having its endpoints in P. The sets A_1, A_2 and A_3 are solid double tori in Int A, also symmetric with respect to P, embedded in A as indicated in Figure 6, where we show the embedding of $A_{n\alpha 1}$, $A_{n\alpha 2}$, $A_{n\alpha 3}$ in $A_{n\alpha}$ for any $n \geqq 0$. Similarly A_{i1}, A_{i2}, and A_{i3} are solid double tori in Int A_i, $(i = 1,2,3)$, embedded in A_i in the same manner as A_1, A_2, A_3 are embedded in A. This process is iterated, and also as in the case of the 2-horned sphere, we set $A_0 = \bigcap_{n=0}^{\infty} (\Sigma A_{n\alpha})$ and let W_{I} be the decomposition of $(E^3 + p)$ consisting

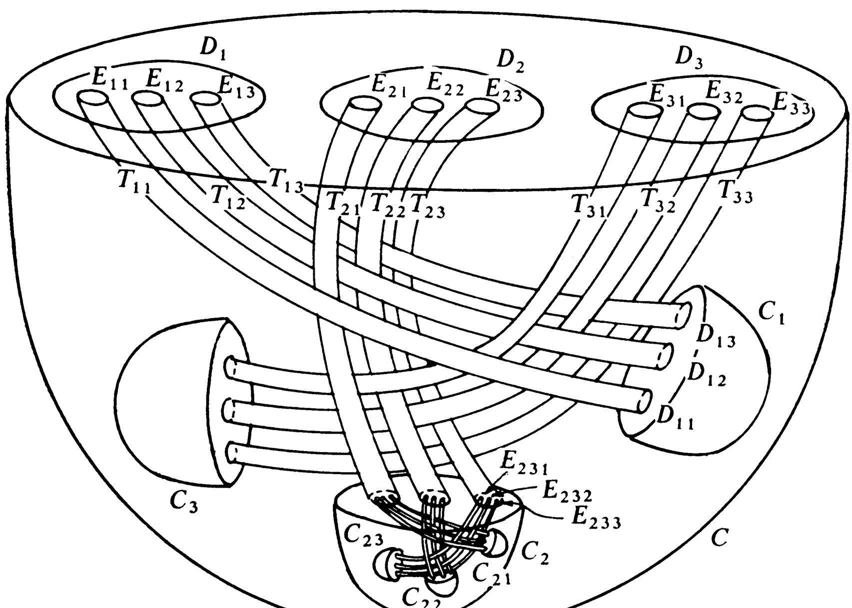

FIGURE 5

of the components of A_0 and the points of $(E^3 + p) - A_0$. We note in passing that this construction may be accomplished in such a way that the components of A_0 are polygonal arcs, in fact the sum of two straight line segments, and such that the angle between these two segments differs from 180° by an arbitrarily small amount.

The set $\mathrm{Cl}(L) - \Sigma \mathrm{Int} A_{1\alpha}$ is homeomorphic with the set $\mathrm{Cl}(\mathrm{Cl}(U) - \Sigma K_{1\alpha})$. This allows us to define homeomorphisms F_1 and G_1 as in §3, and then sequences of homeomorphisms $\{F_n\}$ and $\{G_n\}$ which in turn define a homeomorphism of $\Sigma(M_{\text{I}}^*, M_{\text{I}}^*)$ onto the decomposition space $(E^3 + p)/W_{\text{I}}$.

The plane P intersects each solid double torus $A_{n\alpha}$ in the sum of three disjoint disks. If D is a polyhedral disk in $A_{n\alpha}$ such that $\text{Int}\, D \subset \text{Int}\, A_{n\alpha}$ and there is a homeomorphism of $A_{n\alpha}$ onto itself which takes a component of $P \cap A_{n\alpha}$ onto D, then D is called a *meridional disk* in $A_{n\alpha}$.

A θ-curve is the sum of three *component arcs*, disjoint except for their common endpoints. The two endpoints we call the *junction points* of the θ-curve. We may regard each solid double torus $A_{n\alpha}$ as a neighborhood of a θ-curve which we call the *center* of $A_{n\alpha}$. It will be useful to adopt the terminology of θ-curves for the $A_{n\alpha}$. Thus the *component arcs* of $A_{n\alpha}$ are neighborhoods of the component arcs of the center of $A_{n\alpha}$, and the *junction points* of $A_{n\alpha}$ are the intersections of the component arcs of $A_{n\alpha}$. Similarly we may refer to a *simple closed curve* of $A_{n\alpha}$, the union of two component arcs of $A_{n\alpha}$.

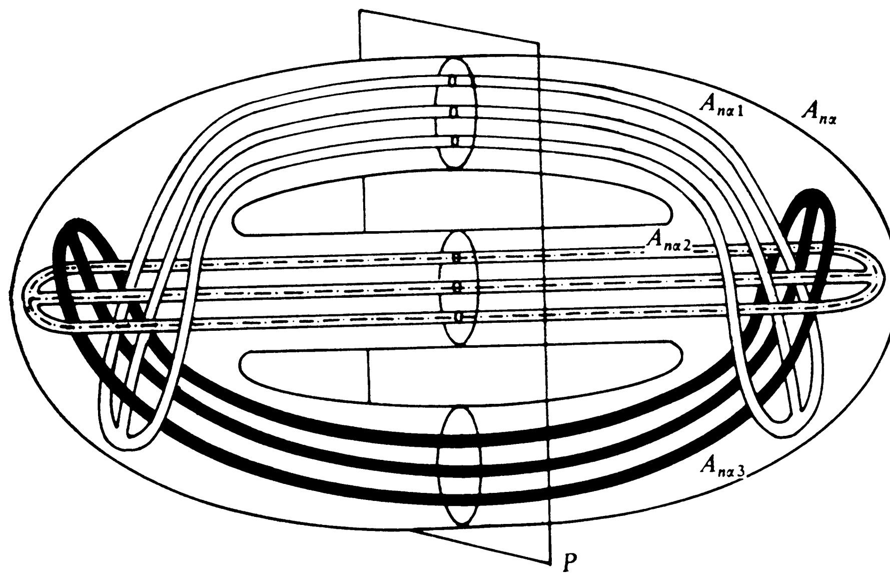

FIGURE 6

Given a simple closed curve J_1 of $A_{n\alpha 1}$, there are two simple closed curves of $A_{n\alpha 2}$ such that if J_2 is either of them, there is a simple closed curve J in $P \cap \text{Bd}\, A_{n\alpha}$ which cannot be shrunk to a point in $A_{n\alpha} - (J_1 \cup J_2)$. Such simple closed curves are said to *link in* $A_{n\alpha}$. However, given any simple closed curve J_1 of $A_{n\alpha 1}$, there is one simple closed curve J_2 of $A_{n\alpha 2}$ such that J_1 and J_2 are not linked in $A_{n\alpha}$. It is this fact that allows us to describe a shrinking process and consequently to show that the decomposition space, and hence $\Sigma(M_I^*, M_I^*)$, is homeomorphic with $E^3 + p$.

LEMMA 2.1. *Suppose P_a, P_b, P_c and P_d are planes parallel to P, ordered from left to right, such that each of the planes intersects the element $A_{n\alpha}$ of $\{A_{n\alpha}\}$*

in three disjoint meridional disks in $A_{n\alpha}$. *Then there is a homeomorphism* h *of* $E^3 + p$ *onto itself such that*

(1) h *is the identity on* $(E^3 + p) - \operatorname{Int} A_{n\alpha}$,

(2) h *is the identity between* P_b *and* P_c, *and*

(3) h *takes each of the twenty-seven components of* $(\Sigma A_{(n+3)\alpha}) \cap A_{n\alpha}$ *into a set which intersects at most one of the two planes* P_a *and* P_d.

Proof. The homeomorphism h is the composition of three homeomorphisms of $E^3 + p$ onto itself. We shall describe these homeomorphisms with the aid of figures; our proof is primarily an explanation of what is pictured in Figures 6 through 11. We make no attempt to give an argument independent of the figures.

A key idea is that of moving a junction point of a solid double torus from its original position to a position between an appropriate pair of planes. Figures 7a and 7b illustrate the effect of such a homeomorphism which moves a junction point of $A_{m\alpha}$ to the right along one component arc as indicated by the arrow 1.

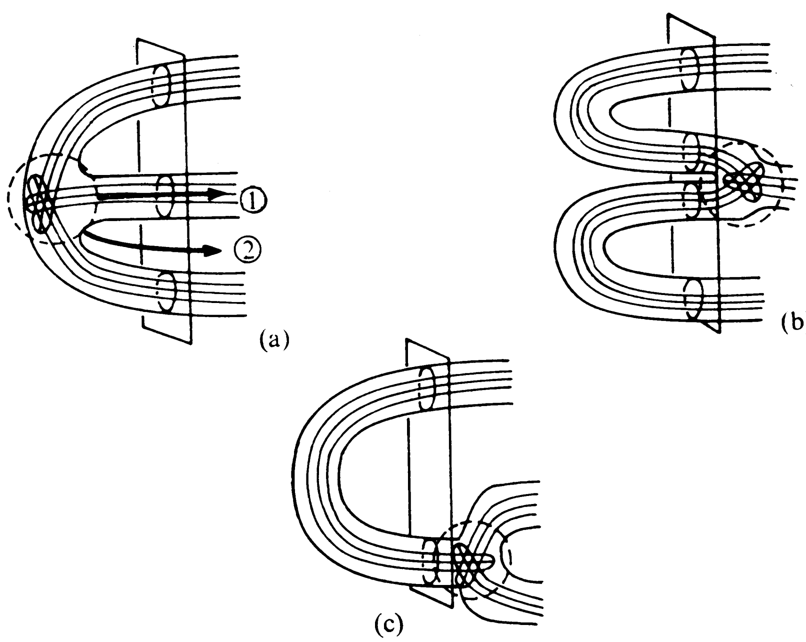

FIGURE 7

It is clear that this can be accomplished by a homeomorphism of $E^3 + p$ onto itself which is the identity outside a small neighborhood of $A_{m\alpha}$. We also imply that the cross sections of the solid double torus $A_{m\alpha}$ remain essentially the same during the homeomorphissm so that the embedding of the three solid double tori at the next stage in $A_{m\alpha}$ is unchanged. This could be made more explicit with a kind of "basic homeomorphism" for double tori, but we feel this to be unnecessary.

Let h_1 be a homeomorphism of $E^3 + p$ onto itself which is fixed outside $A_{n\alpha}$ and between P_b and P_c and which moves the left junction point of each $A_{n\alpha i}$ to the right along one component arc to a position between planes P_a and P_b, $(i = 1,2,3)$. The homeomorphism h_1 also moves the right junction point of each $A_{n\alpha i}$ to the left along a different component arc to a position between planes P_c and P_d. Figure 8 shows the result of applying h_1 and may be compared with Figure 6 even though planes P_a, P_b, P_c and P_d are not shown in Figure 6. It may be observed that the embeddings of $h_1(A_{n\alpha 1})$, $h_1(A_{n\alpha 2})$, and $h_1(A_{n\alpha 3})$ are all the same relative to the planes P_a, P_b, P_c, and P_d. This allows us to describe the second homeomorphism h_2 in terms of its action on any $h_1(A_{n\alpha i})$, $(1 \leqq i \leqq 3)$.

One component arc of $h_1(A_{n\alpha i})$ intersects P_a but not P_d, one component arc intersects P_d but not P_a, and one component arc intersects both P_a anb P_d. Each intesection of a component arc of $h_1(A_{n\alpha i})$ with P_a or P_d is a meridional disk in $h_1(A_{n\alpha i})$. As may be seen from Figure 8, $h_1(A_{n\alpha i})$ is rather convoluted relative

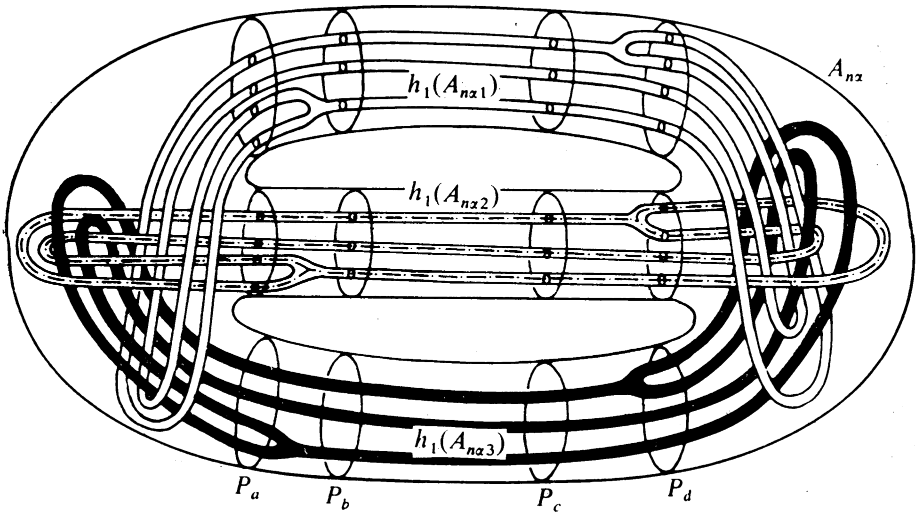

FIGURE 8

to P_a and P_d. In order to show more clearly the effect of h_2 we have drawn $h_1(A_{n\alpha i})$ in Figure 9 with the same shape as that of $A_{n\alpha}$ in Figure 6.This of course results in some distortion of the planes P_a and P_d, so we have simply represented the components of the intersections of P_a and P_d with $h_1(A_{n\alpha i})$. It should be emphasized again that Figure 9 is really a schematic representation of one of the $h_1(A_{n\alpha i})$ in Figure 8. While the meridional disks of $P_a \cap h_1(A_{n\alpha i})$ and $P_d \cap h_1(A_{n\alpha i})$ appear to be closer together on the respective component arcs in Figure 9 than they do in Figure 8, the important fact is that they are consecutive on each component arc, i.e., no meridional disk from any other plane is between two components of $P_a \cap h_1(A_{n\alpha i})$ on a given component arc of $h_1(A_{n\alpha i})$, and similarly for P_d.

The homeomorphism h_2 agrees with h_1 outside $h_1(A_{nai})$ and is the identity between P_b and P_c. It moves the left junction points of $h_1(A_{nai1})$ and $h_1(A_{nai2})$ to the right along *simple closed curves* of $h_1(A_{nai1})$ and $h_1(A_{nai2})$, respectively, to a position between the planes P_a and P_b. This type of homeomorphism is illustrated in Figures 7a and 7c where the junction point is moved along arrow 2. Since h_2 in effect "shortens" one end of a given simple closed curve by reducing the number of planes it intersects, the simple closed curves which link the given one must be "stretched" in the process. This is shown in Figure 9. Similarly h_2 moves the right junction points of $h_1(A_{nai2})$ and $h_1(A_{nai3})$ to the left to a position between planes P_c and P_d as shown in the right portion of Figure 9.

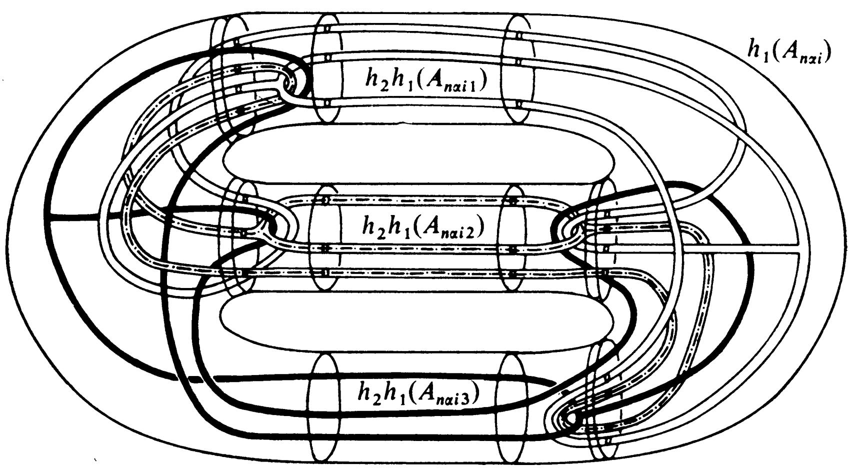

FIGURE 9

It may be helpful here to examine the intersections of the component arcs of the $h_2h_1(A_{n\alpha ij})$, $(i, j = 1, 2, 3)$, with the end planes P_a and P_d. Each component is a meridional disk in the appropriate solid double torus. Furthermore, no meridional disk from any other plane is between two components of $P_a \cap h_2h_1(A_{n\alpha ij})$ on one component arc of $h_2h_1(A_{n\alpha ij})$, and similarly for $P_d \cap h_2h_1(A_{n\alpha ij})$. This allows us to treat the intersection of P_a or P_d with a given component arc as a single disk. Then, at a later stage, when we wish to move a junction point to a position between a given pair of planes, we move through all the components of the intersection of P_a or P_d with given component arc. The intersections of P_a and P_d with the $h_2h_1(A_{n\alpha ij})$ are not the same for $j = 1,2,3$. Each of the sets $h_2h_1(A_{n\alpha i1})$ and $h_2h_1(A_{n\alpha i3})$ has one component arc intersecting both P_a and P_d, one component arc intersecting only one of P_a and P_d, and one component arc which does not intersect either. The set $h_2h_1(A_{n\alpha i2})$ has one component arc which intersects each of P_a and P_d in eight meridional disks and two component arcs which fail to intersect either P_a or P_d. The intersections are represented schematically in Figure 10. We have kept the same order for the components arcs (from top to bottom) in Figure 10 as appears in Figure 9 for the reader's convenience in comparing the figures. Our pictures have the effect of magnifying the ends of the solid double tori, but everything that occurs to the left of P_b, for example, in Figure 9 is a subset of the portion of $A_{n\alpha}$ to the left of P_b in Figure 8. If this portion of $A_{n\alpha}$ is of small diameter to begin with, all of the stretching in Figures 8 and 9 takes place in this small set. While the "arc length" of some of these sets may increase, the diameters of the sets are decreasing.

Now we are in a position to describe the third homeomorphism h_3. Again, h_3 agrees with h_2 outside $h_2h_1((\sum A_{(n+2)\alpha}) \cap (A_{n\alpha}))$, and h_3 is fixed between the planes P_b and P_c. The effect of h_3 is to shorten one component of $h_2h_1(\sum A_{(n+3)\alpha})$ in each of the sets $h_2h_1(A_{n\alpha ij})$, $(i,j = 1,2,3)$. The appropriate component is indicated by the arrows in Figure 10. For example, h_3 pulls the right junction point of $h_2h_1(A_{n\alpha i1})$ to the left until it is between planes P_c and P_d as shown in Figure 11. A picture similar to the one shown in Figure 11 could be given for each of the cases illustrated in Figure 10.

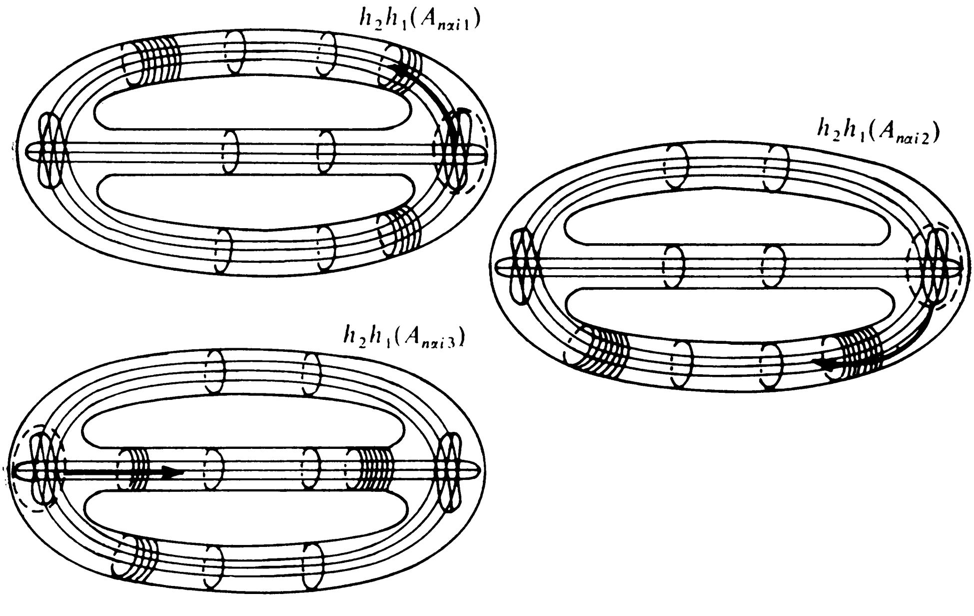

FIGURE 10

The homeomorphism h of the lemma is $h_3h_2h_1$.

LEMMA 2.2. *Given an $\varepsilon > 0$ and a positive integer j, there is an integer s and a homeomorphism h of $E^3 + p$ onto itself which is fixed on $(E^3 + p) - \sum A_{j\alpha}$ and which takes each component of $\sum A_{s\alpha}$ onto a set of diameter less than ε.*

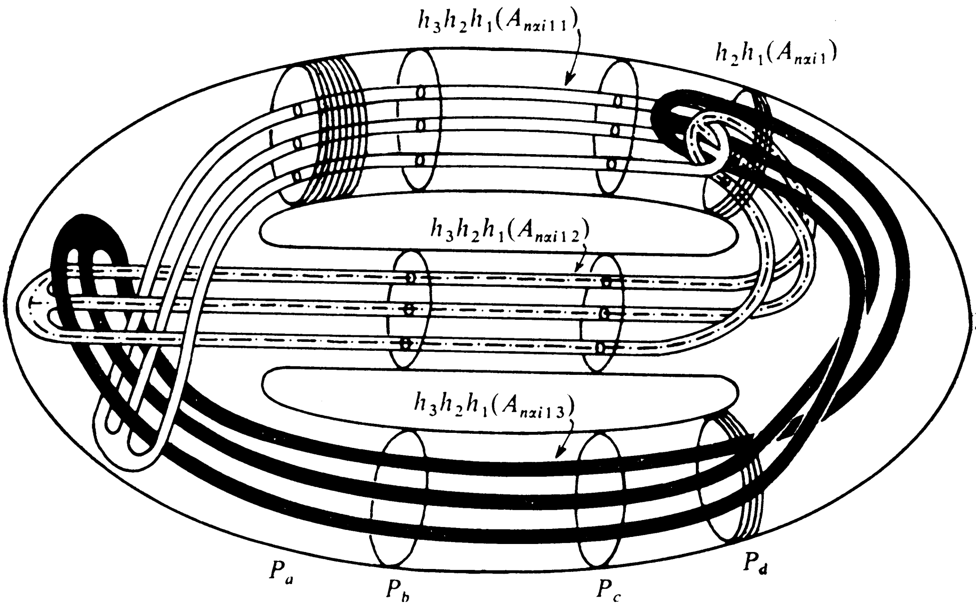

FIGURE 11

Proof. This is proved by repeated applications of Lemma 2.1. Given ε, there is an integer n greater than j and a finite sequence of planes $P_1, P_2, \cdots, P_k$, ordered from left to right, parallel to P, intersecting each component of $\Sigma A_{m\alpha}$ $(m = 0, 1, 2, \cdots)$ in three meridional disks, such that each component of $(\Sigma A_{n\alpha}) - (\bigcup_1^k P_i)$ has diameter less than $\varepsilon/4$.

Applying Lemma 2.1, there is a homeomorphism h_1 of $E^3 + p$ onto itself which is fixed outside $\Sigma A_{n\alpha}$ and between P_2 and P_{k-1} and which takes each component of $\Sigma A_{(n+3)\alpha}$ into a set which intersects at most one of the planes P_1 and P_k.

Suppose a component $h_1(A_{(n+3)\alpha})$ of $\Sigma h_1(A_{(n+3)\alpha})$ intersects only $k-1$ of the planes $P_1, P_2, P_3, \cdots, P_k$. As observed in the proof of Lemma 2.1, P_1 or P_k may contain several meridional disks in the component of $\Sigma h_1(A_{(n+2)\alpha})$ containing $h_1(A_{(n+3)\alpha})$. However, all meridional disks from P_1 or P_k occur consecutively on any one component arc. To apply Lemma 2.1 again, we may choose the first such disk to the left of P_2, relative to $h_1(A_{(n+3)\alpha})$, for P_a in the statement of Lemma 2.1. Alternatively we may think of the intersections with P_1 as a single disk, as mentioned above. The planes P_2, P_{k-2}, and P_{k-1} are used for P_b, P_c, and P_d, respectively, in this case. The procedure is similar if $h_1(A_{(n+3)\alpha})$ intersects only $P_2, P_3, \cdots, P_k$, where we use P_2, P_3, P_{k-1}, and P_k for the planes of Lemma 2.1.

Iterating this process, in a finite number, say v, of steps we reach a stage at which no solid double torus intersects four of the planes $P_1, P_2, \cdots, P_k$. If we denote this stage by s and the composition of the homeopmorphisms $h_1, h_2, \cdots, h_v$ by h, then h is fixed outside $h_1(\Sigma A_{(n+1)\alpha})$ so each component of $h(\Sigma A_{s\alpha})$ is a subset of one component arc of $A_{n\alpha}$. It follows that each component of $h(\Sigma A_{s\alpha})$

intersects at most three components of $(\sum A_{n\alpha}) - (\bigcup_1^k P_i)$ and hence has diameter less than ε.

THEOREM 2. *The continuum* $\Sigma(M_{\mathrm{I}}^*, M_{\mathrm{I}}^*)$ *is homeomorphic with* S^3.

Proof. This theorem follows from the preceding lemma in the same manner as Theorem 1 follows from Lemma 1.2.

5. **The 3-horned sphere M_{II}.** In this section we describe an example which is illustrative of a class of solid 3-horned spheres which do not yield S^3 even when sewn together with the identity homeomorphism.

For some time we conjectured that $\Sigma(M_{\mathrm{I}}^*, M_{\mathrm{I}}^*)$ discussed in §4 was topologically different from S^3. There are several examples in the literature of point-like decompositions of E^3 defined by the intersection of cells with handles, for instance [5], [7], [9], [12]. One of the criteria for determining whether the associated decomposition space is or is not E^3, for the examples with which we were familiar, seemed to be whether or not a shrinking process could be begun by some effective shrinking of the defining sets of one stage in the sets of the preceding stage. That this is not sufficient is indicated in §4 where three stages are required to accomplish any significant shrinking.

The horned sphere M_{I} is the 3-horned analogue of Alexander's 2-horned sphere. There is a 4-horned analogue which also has the property that the sum of two sewn together with the identity homeomorphism is topologically S^3 but for which no significant shrinking can be done in less than four stages. The techniques involved are very similar to those used in §4, and we do not describe the example here. There are k-horned analogues of Alexander's 2-horned sphere for every integer greater than 2, and it seems entirely possible that the sum of two such solid horned spheres, under the identity homeomorphism, is homeomorphic with S^3. As yet we have seen no satisfactory way to extend our methods to this general case. The methods used below to show that $\Sigma(M_{\mathrm{II}}^*, M_{\mathrm{II}}^*)$ is not E^3, however, can be used to show that for each integer k greater than 2 there is a solid k-horned sphere such that S^3 does not result from sewing two of them together along their boundaries with the identity homeomorphism.

The description of the horned sphere M_{II} is so much like that of M_{I} that we do not give details. We picture the initial stages of the construction of M_{II} in Figure 12 and show the first stage of the construction of the associated decomposition of $E^3 + p$ in Figure 13. In Figure 13, A_1, A_2, and A_3 are shown as θ-curves but each is actually a solid double torus. A comparison of these figures with Figures 5 and 6 should make the procedure clear.

We observe that this construction can be done so that the nondegenerate elements of the decomposition W_{II} associated with $\Sigma(M_{\mathrm{II}}^*, M_{\mathrm{II}}^*)$ are tame; arcs which are locally polyhedral except at their endpoints. However, it does not appear that they can be made polyhedral. Consequently it seems that Fort's

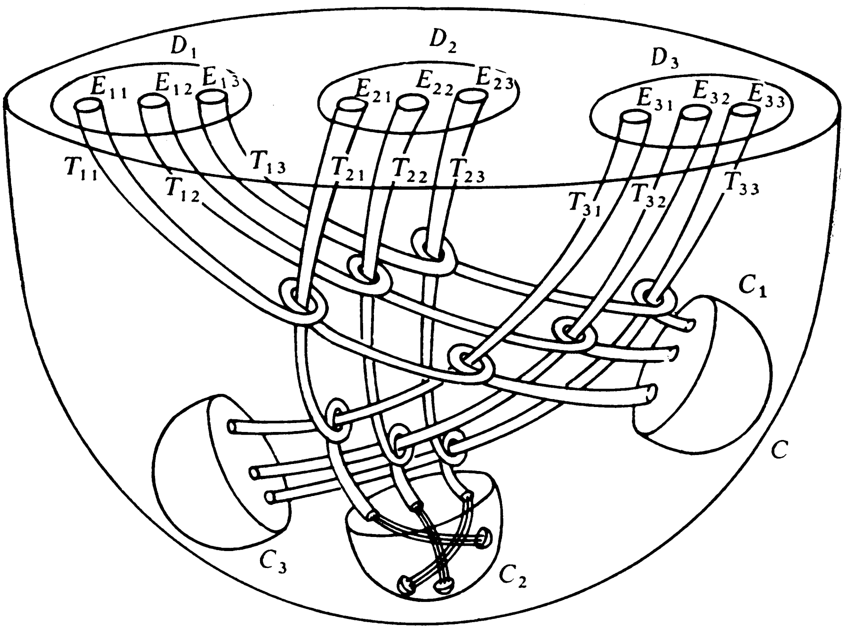

FIGURE 12

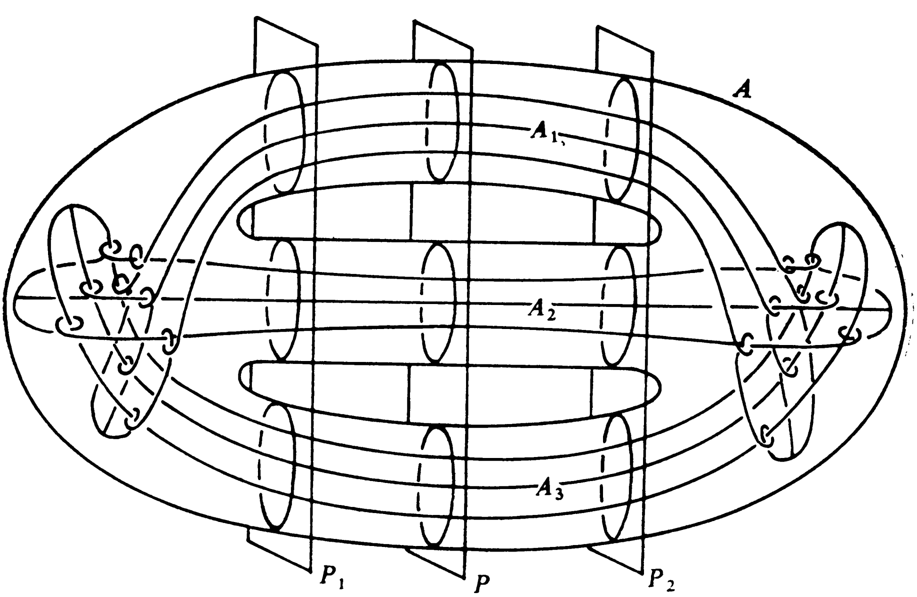

FIGURE 13

modification of Bing's "dogbone" construction [12] is still the closest example to a decomposition of E^3 into points and straight line segments such that the decomposition space is not E^3. Bing has described a decomposition into points and straight line segments [9], but it is not yet known whether it the decomposition space is E^3. McAuley announced such an example [13], but the announced example needed to be modified, and as far as we are aware, a proof has not yet been given.

An important distinction between the construction of M_{I} and the construction of M_{II} is the additional linking that is introduced. The construction of M_{II} guarantees that each simple closed curve of A_1, for example, links *all three* simple closed curves of each of A_2 and A_3 in A. It will become clear in the proof of Theorem 4 that many similar examples may be constructed. It seems that M_{II} is in some ways the "simplest" horned sphere such that the sum of two sewn together with the identity homeomorphism is not homeomorphic with S^3.

Having a simple closed curve link all three simple closed curves of a θ-curve is not possible using the theory of linking simple closed curves as developed by Bing [6, §9], because his Theorem 9 states that a simple closed curve links exactly two simple closed curves of a θ-curve if it links one of them. Bing's exposition is really a simple description, shorn of algebraic terminology, of homological linking modulo 2, that is, using Z_2 as coefficient group and ignoring problems of orientation. He observes that if the coefficient group is not Z_2, then some changes must be made in the statement of two of his theorems. While a simplified development of homological linking can be given in a manner similar to Bing's, we do not feel it to be necessary in the light of his observations. A complete development is available in [1, Chapter XV].

We restate two of Bing's theorems from [6] so that "exactly two" in his statements is replaced with "at least two," where it is understood that we are using the more general linking mentioned above.

THEOREM 9 (BING). *If a polygonal simple closed curve J does not intersect a certain polygonal θ-curve, then J links at least two simple closed curves in the θ-curve if it links one of them.*

THEOREM 9′ (BING). *Suppose f is the map of a θ-curve T into E^3 where f_1, f_2, f_3 are the parts of f operating on the simple closed curves in T. If f_4 is a map of a circle J into E^3 such that $f_4(J) \cap f(T) = \varnothing$, then f_4 links at least two of f_1, f_2, f_3 if it links one of them.*

Our purpose for including these theorems is the observation that Bings's Theorem 3 of [7] still holds if we interpret "linking" in the statement of his Theorem 3 to be homological linking. The only use he makes of Theorems 9 and 9′ in the proof of Theorem 3 does not depend on the word "exactly."

Our proof that the decomposition space associated with $\Sigma(M_{\text{II}}^*, M_{\text{II}}^*)$ is not homeomorphic with S^3 is very similar to that used by Bing in [7]. We will make an effort to make our notation consistent with his. For this reason it is just as easy to consider the decomposition W_{II} to be of E^3 rather than S^3, and we will show that the decomposition space is not homeomorphic with E^3.

We regard the double torus $A_{n\alpha}$ as a neighborhood of a θ-curve which we call the *center* of $A_{n\alpha}$. If $h(A_{n\alpha})$ is the image of $A_{n\alpha}$ under a homeomorphism h and $F_{n\alpha}$ is the center of $A_{n\alpha}$, we call $h(F_{n\alpha})$ the *center* of $h(A_{n\alpha})$. Any θ-curve in $h(A_{n\alpha})$ which is homotopic in $h(A_{n\alpha})$ to the center of $h(A_{n\alpha})$ will be called a *central θ-curve* of $h(A_{n\alpha})$.

In the discussion that follows, P_1 and P_2 will be vertical planes such that each P_i intersects A in the sum of three disjoint disks as indicated in Figure 13.

Property $P(k)$. A θ-curve has Property $P(k)$ if each of k simple closed curves in the θ-curve intersects both P_1 and P_2, $(0 \leqq k \leqq 3)$.

Property $Q(k)$. An image of A under a homeomorphism h has Property $Q(k)$ if each of its central θ-curves has Property $P(k)$.

THEOREM 3. *Suppose that h is a homeomorphism of E^3 onto itself which is fixed on $E^3 - \operatorname{Int} A$. Then $h(A)$ has Property $Q(3)$.*

Proof. If F is the center of A, then $h(F)$, the center of $h(A)$, is homotopic in A to F because there is a θ-curve on $\operatorname{Bd} A$ which is homotopic in A to F. Let F' be any central θ-curve in $h(A)$. Since homotopy is transitive, any such F' is a central θ-curve in A.

Each of the planes P_1 and P_2 intersects A in the sum of three disjoint disks. Let D_{i1}, D_{i2}, and D_{i3} be disjoint disks in P_i $(i = 1, 2)$ each containing in its interior one of the disks in the the intersection of P_i with A. Since each simple closed curve of F links the boundaries of two of D_{11}, D_{12}, D_{13} and of two of D_{21}, D_{22}, D_{23}, the same thing is true of each simple closed curve of F'. It follows then that each simple closed curve in F' intersects both P_1 and P_2.

THEOREM 4. *If B is the image of A under a homeomorphism h of E^3 onto itself and B has Property $Q(3)$, then one $h(A_1), h(A_2), h(A_3)$ also has Property $Q(3)$.*

Proof. Suppose that no $h(A_i)$, $(1 \leqq i \leqq 3)$, has Property $Q(3)$. Then each A_i contains a central θ-curve F_i with the property that F_i contains a simple closed curve J_i such that $h(J_i)$ intersects at most one of the planes P_1 and P_2.

Let V be the universal covering space of A, a portion of which is represented in Figure 14b, and let f be the projection map of V onto A. We may also think of the portion of V shown in Figure 14b as an "abstract cube" similar to the one used by Bing in the proof of Theorem 10 of [7]. Each J_i may be shrunk to a point in Int A, so $f^{-1}(J_i)$ is an infinite collection of disjoint simple closed curves.

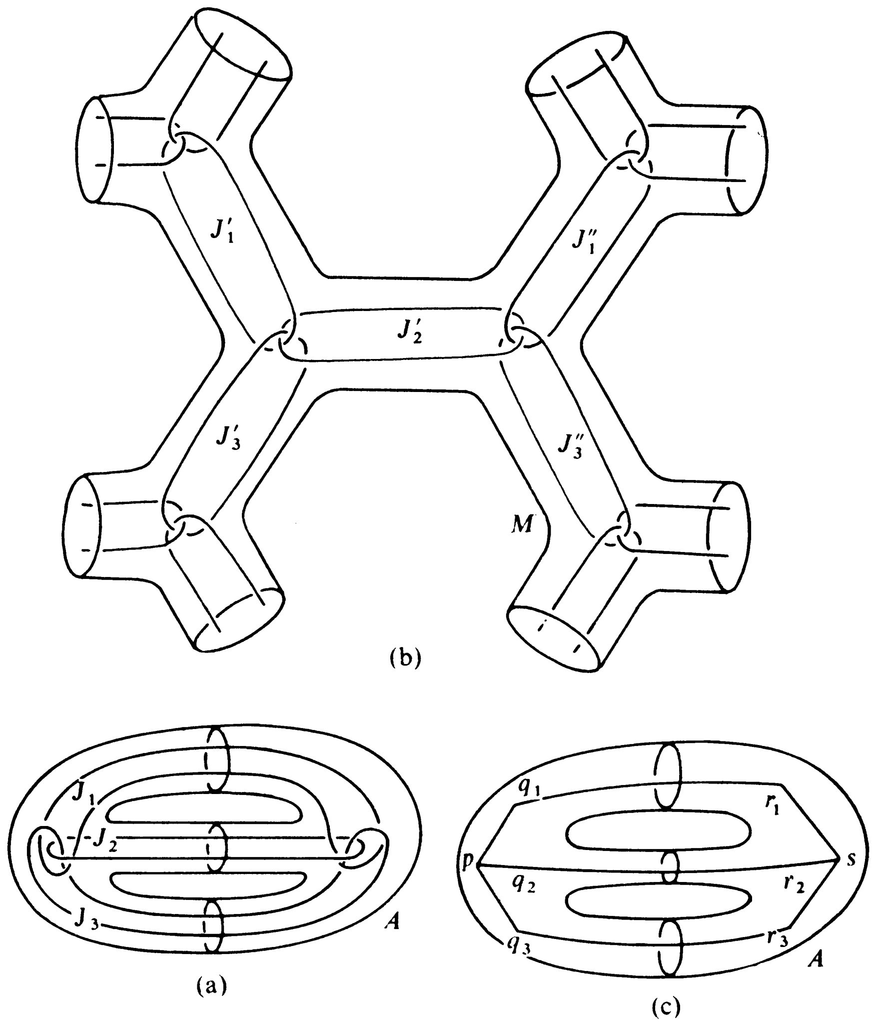

FIGURE 14

in Int V. We let J_1', J_2', J_3', J_1'', J_3'' be components of $f^{-1}(J_i)$ (various "copies" of the J_i) in V, such that each pair of J_1', J_2', J_3' is linked, and each pair of J_1'', J_2', J_3'' is linked. That this may be done follows from the observation made above in our description of this example that each simple closed curve of A_i links all three simple closed curves of A_j in A $(1 \leqq i,j \leqq 3, i \neq j)$. While some simple closed curves of the A_i that link in A may not be linked in E^3, appropriate copies under f^{-1} are actually linked. Three essentially different types of linking occur in V and we have represented these types in Figure 15. For the argument we use, however, the type of linking is not important. What is important is that each pair be linked. This is the justification for the statement made above that M_1

is representative of a whole class of 3-horned spheres differing in construction only enough to change the linking appearing in the decomposition of E^3. Because of this observation, and to enable us to use a simpler picture that may be more easily understood, we show only the simplest kind of linking for J_1, J_2, J_3 in A and copies of J_1, J_2, J_3 in V in Figure 14. The actual situation resulting from the linking indicated in Figure 13 would be much more complicated.

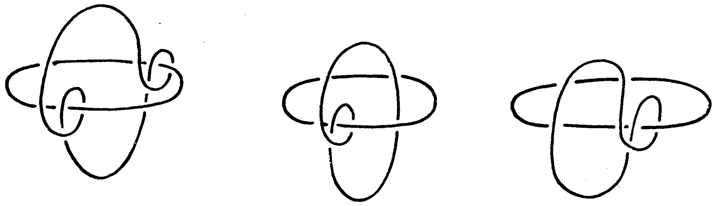

FIGURE 15

Since each component of $A \cap h^{-1}(P_1 \cup P_2)$ is a tame disk, each component of $f^{-1}(A \cap h^{-1}(P_1 \cup P_2))$ is a tame disk. Those components of $f^{-1}(A \cap h^{-1}(P_1 \cup P_2))$ which intersect $(J'_1 \cup J'_2 \cup J'_3 \cup J''_1 \cup J''_3)$ may be considered subsets of a single tame disk D, and we may assume without loss of generality that no copy of any J_1, J_2, J_3 is contained in D.

Since V is in E^3, it follows from Theorem 3 of [7] that for each pair of elements of J'_1, J'_2, J'_3 there is a component of (Int V) $- D$ which intersects both of these elements. Then, by Theorem 5 of [7] since the universal covering in space is unicoherent, there is a component of (Int V) $- D$ which intersects each of J'_1, J'_2, J'_3. Hence there are arcs $p'q'_i$ in (Int V) $- D$, running from a point p' to point q'_i in J'_i $(i=1,2,3)$ which are disjoint except for their common endpoint p'. We may also require that the projections of these arcs be arcs pq_i which are disjoint except for p and such that no pq_i intersects $h^{-1}(P_1 \cup P_2)$ and no pq_i intersects $\bigcup_{j=1}^{3} J_j$ except in the point q_i. Similarly we may get arcs r_is, $(i = 1, 2, 3)$ from points r_i in J_i to a point s in A such that

$$(r_is) \cap (r_js) = s \text{ if } i \neq j,\ (r_is) \cap (\bigcup_{j=1}^{3} J_j) = r_i,\ (\bigcup_{i=1}^{3} pq_i) \cap (\bigcup_{j=1}^{3} r_js) = \varnothing$$

and $(r_is) \cap (h^{-1}(P_1 \cup P_2)) = \varnothing$. Since no one of $h(J_1)$, $h(J_2)$, $h(J_3)$ intersects both P_1 and P_2, there is an arc in J_i from q_i to r_i which misses at least one of $h^{-1}(P_1)$ and $h^{-1}(P_2)$. Then $\bigcup_{i=1}^{3} pq_ir_is$ is a θ-curve H in A which is homotopic in A to the central θ-curve of A indicated in Figure 14c. To see that this is so we simply observe that if p' and q' are points of V and $p'x'q'$ and $p'y'q'$ are arcs in V from p' to q', then $p'x'q'$ and $p'y'q'$ are homotopic in Int V under a homotopy which is fixed on p' and q'; otherwise V would not be simply connected.

Each component arc of H fails to intersects one of $h^{-1}(P_1)$ and $h^{-1}(P_2)$ so at least two component arcs of H miss the same one of $h^{-1}(P_1)$ and $h^{-1}(P_2)$. The sum of these two component arcs is a simple closed curve K in H and $h(K)$

is a simple closed curve in a central θ-curve of B which does not intersect both P_1 and P_2, contrary to the hypothesis that B has Property $Q(3)$.

The following theorem is an immediate consequence of Theorems 3 and 4. We state it primarily for reference for our final result stated in Theorem 6. Compare [7, Theorem 11].

THEOREM 5. *If h is any homeomorphism of E^3 onto itself which is fixed on $E^3 - \text{Int } A$, then there is an element w of the decomposition W_{11} such that $h(w)$ intersects both P_1 and P_2.*

THEOREM 6. *The decomposition space E^3/W_{11} associated with $\Sigma(M_{11}^*, M_{11}^*)$ is not homeomorphic with E^3.*

Proof. This theorem follows from Theorem 5 in almost exactly the same way as Theorem 12 of [7] follows from Theorem 11 of that paper. Alternatively, we may use a theorem recently announced by Armentrout [3] to the effect that if the nondegenerate elements of an upper semicontinuous point-like decomposition G of E^3 form a Cantor set in the decomposition space and E^3/G is homeomorphic with E^3, then for any $\varepsilon > 0$ and for any open set U containing all the nondegenerate elements of G there is a homeomorphism of E^3 onto itself which is fixed outside U and which takes every element of G onto a set of diameter less than ε. By Theorem 5, the existence of such a homeomorphism is impossible for the decomposition W_{11}.

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